Defining gravitational charges using spin-2 field and CYK tensors

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Abstract

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The goal is to define all 20 gravitational charges (momentum p_μ, angular momentum $j_{\mu\nu}$ and duals) using conformal Yano-Killing tensors (CYK tensors) and spin-2 field (Weyl tensor) for Minkowski spacetime.
This approach is analogous to the one used in classical electrodynamics and in the end we show that the contractions of a Weyl tensor with CYK tensors have to fulfill equations similar to vacuum Maxwell equations for electromagnetic tensor $F^{\mu\nu}$.
Afterwards we use $3+1$ decomposition to extend this method to other spacetimes which are spatialy-conformally-flat (e.g. Schwarzschild metric).

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Introduction

Spin-2 field

Definition

Tensor field $W_{\alpha\beta\mu\nu}$ is a spin-2 field if and only if when the following occurs:

$$\text{algebraic:} \begin{cases} W_{\alpha\beta\mu\nu} = W_{\mu\nu\alpha\beta} = W_{[\alpha\beta][\mu\nu]} \\ W_{\alpha[\beta\mu\nu]} = 0 \qquad g^{\alpha\mu} W_{\alpha\beta\mu\nu} = 0 \end{cases}$$

$$\text{differential:} \quad \nabla_{[\lambda} W_{\alpha\beta]\mu\nu} = 0 \quad \text{(field equation)}$$

Dual field:

$$^*W_{\alpha\beta\mu\nu} := \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\rho\sigma} W_{\rho\sigma\mu\nu}$$

$$W^*{}_{\alpha\beta\mu\nu} := \frac{1}{2} W_{\alpha\beta\rho\sigma}\varepsilon^{\rho\sigma}{}_{\mu\nu}$$

Electric and magnetic part

$$E_{\lambda\nu} := W_{\lambda\mu\nu\rho} n^\mu n^\rho$$

$$H_{\lambda\nu} := W^*{}_{\lambda\mu\nu\rho} n^\mu n^\rho$$

where n^μ is a normed vector field normal to foliation Σ_t.

Properties

Spin-2 field (only for 4 dimmensional space)

$$^*W = W^* \qquad {}^*({}^*W) = {}^*W^* = W$$

$$\nabla_{[\lambda} W_{\alpha\beta]\mu\nu} = 0 \iff \nabla^{\alpha} W_{\alpha\beta\mu\nu} = 0 \iff$$
$$\iff \nabla_{[\lambda} {}^*W_{\alpha\beta]\mu\nu} = 0 \iff \nabla^{\alpha *} W_{\alpha\beta\mu\nu} = 0$$

Electric and magnetic part

- $E_{\mu\nu}$ and $H_{\mu\nu}$ are symmetric and traceless,
- $E_{\lambda\nu} n^{\nu} = H_{\lambda\nu} n^{\nu} = 0$.

Introduction
CYK tensors

Definition

Antisymmetric tensor $Q_{\mu\nu}$ is a conformal Yano-Killing tensor (CYK tensor) for a metric g, iff $\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) = 0$.

Where:

$$\mathcal{Q}_{\lambda\kappa\sigma}(Q, g) := Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} - \frac{2}{3}(g_{\sigma\lambda}Q^{\nu}{}_{\kappa;\nu} + g_{\kappa(\lambda}Q_{\sigma)}{}^{\mu}{}_{;\mu})$$

Theorem

If $Q_{\mu\nu}$ is a CYK tensor for a metric $g_{\mu\nu}$, then $\Omega^3 Q_{\mu\nu}$ is a CYK tensor for a metric $\Omega^2 g_{\mu\nu}$.

Theorem

Antisymmetric tensor $Q_{\mu\nu}$ is a CYK tensor of a four-dimmensional metric $g_{\mu\nu}$ iff when dual tensor $*Q_{\mu\nu}$ is a CYK tensor for the same metric g.

Introduction

Let $W_{\alpha\beta\mu\nu}$ to be a spin-2 field and $Q_{\mu\nu}$ to be any antisymmetric tensor.

$$F_{\mu\nu}(W, Q) := W_{\mu\nu\alpha\beta} Q^{\alpha\beta}$$

Lemma

$$W^{\mu\nu\alpha\beta} \mathcal{Q}_{\alpha\beta\nu} = \frac{3}{2} W^{\mu\nu\alpha\beta} Q_{\alpha\beta;\nu}$$

$$\begin{aligned} \nabla_\nu F^{\mu\nu}(W, Q) &= \nabla_\nu (W^{\mu\nu\alpha\beta} Q_{\alpha\beta}) = (\nabla_\nu W^{\mu\nu\alpha\beta}) Q_{\alpha\beta} + W^{\mu\nu\alpha\beta} (\nabla_\nu Q_{\alpha\beta}) \\ &= \frac{2}{3} W^{\mu\nu\alpha\beta} \mathcal{Q}_{\alpha\beta\nu} \end{aligned}$$

If $Q_{\mu\nu}$ is a CYK tensor then $\mathcal{Q}_{\alpha\beta\nu}(Q, g) = 0$.

Conclusion

Let V be a 3-volume and ∂V be its boundary, $W_{\alpha\beta\mu\nu}$ is a spin-2 field, $Q_{\mu\nu}$ CYK tensor then:

$$\int_{\partial V} F^{\mu\nu}(W, Q) d\sigma_{\mu\nu} = \int_V \nabla_\nu F^{\mu\nu}(W, Q) d\Sigma_\mu = 0$$

Divergence theorem

Theorem

Let $W_{\alpha\beta\mu\nu}$ be a spin-2 field, $Q_{\mu\nu}$ CYK tensor, then antisymmetric tensor $F_{\mu\nu} = W_{\mu\nu\alpha\beta}Q^{\alpha\beta}$ fulfills vacuum Maxwell equations:

$$\nabla_\lambda F^{\mu\lambda} = \nabla_\lambda * F^{\mu\lambda} = 0$$

where $*F^{\mu\lambda} = \frac{1}{2}\varepsilon^{\mu\lambda}{}_{\rho\sigma}F^{\rho\sigma}$.

Conclusion

The theorem states that the flow of a tensor $W_{\alpha\beta\mu\nu}Q^{\mu\nu}$ through any closed 2-dimmensional surfaces V_1 and V_2 is equal if exists a 3-volume V between them ($\partial V = V_1 - V_2$). The value of this flow is called the charge coresponding to the tensor $Q_{\mu\nu}$.

CYK tensors for Minkowski

Base of solutions of the equation $\mathcal{Q}_{\mu\nu\lambda} = 0$ (for $\mu < \nu$)

$$\mathcal{T}_\mu \wedge \mathcal{T}_\nu, \quad \mathcal{D} \wedge \mathcal{T}_\mu, \quad *(\mathcal{D} \wedge \mathcal{T}_\mu), \quad \mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu$$

where: $\mathcal{D} := x^\mu \partial_\mu, \quad \mathcal{T}_\mu := \partial_\mu, \quad \mathcal{L}_{\mu\nu} := x_\mu \partial_\nu - x_\nu \partial_\mu$

Then for any tensor:

$$Q = \tilde{q}^{\mu\nu}\mathcal{T}_\mu \wedge \mathcal{T}_\nu + \tilde{u}^\mu \mathcal{D} \wedge \mathcal{T}_\mu + \tilde{v}^\mu * (\mathcal{D} \wedge \mathcal{T}_\mu) + \tilde{k}^{\mu\nu}(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu)$$

for constant, antisymmetric tensors $\tilde{q}^{\mu\nu}$, $\tilde{k}^{\mu\nu}$ and constant vectors $\tilde{u}^\mu$, $\tilde{v}^\mu$.

Another base (3+1 decomposition)

$$\mathcal{T}_0 \wedge \mathcal{T}_k, \quad \mathcal{T}_0 \wedge \mathcal{D}, \quad \mathcal{T}_k \wedge \mathcal{D}, \quad \mathcal{D} \wedge \mathcal{L}_{0k} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_0 \wedge \mathcal{T}_k$$

And dual tensors to ones listed above.

Charges associated with base tensors

Charges

$$w_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} W(\mathcal{T}_\mu \wedge \mathcal{T}_\nu) \quad \stackrel{!}{=} 0$$

$$p_\mu := \frac{1}{16\pi}\int_{\partial\Sigma} W(\mathcal{D} \wedge \mathcal{T}_\mu) \quad \to \text{momentum}$$

$$b_\mu := \frac{1}{16\pi}\int_{\partial\Sigma} {}^*W(\mathcal{D} \wedge \mathcal{T}_\mu) \quad \to \text{dual momentum}$$

$$j_{\mu\nu} := \frac{1}{16\pi}\int_{\partial\Sigma} W(\mathcal{D} \wedge \mathcal{L}_{\mu\nu} - \frac{1}{2}\eta(\mathcal{D},\mathcal{D})\mathcal{T}_\mu \wedge \mathcal{T}_\nu) \quad \to \text{angular momentum tensor}$$

where $W(A \wedge B) := W^{\mu\nu}{}_{\alpha\beta} A^\alpha B^\beta d\sigma_{\mu\nu}$

Schwarzschild metric

The metric

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Let's define $\bar{r}$ so that: $r = \bar{r}(1 + \frac{M}{2\bar{r}})^2$.

$$\Downarrow$$

$$g = -\left(\frac{1 - M/2\bar{r}}{1 + M/2\bar{r}}\right)^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 \left[d\bar{r}^{\,2} + \bar{r}^{\,2}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

Conclusion

For t=*const* Schwarzschild metric is conformaly flat.
We have all 10 conformal Killing vectors.

How to define charges?

Observation

Every CYK tensor $Q_{\mu\nu}$ for Minkowski metric can be written in a following form:

$$Q = a(t)\mathcal{T}_0 \wedge X + b(t) * (\mathcal{T}_0 \wedge Y)$$

where X, Y are conformal Killing vectors (3-dimmensional).

Summary

- Spin-2 field and CYK tensors can be used to define gravitational charges in a way similar to the one used in classical electromagnetism,
- We have 20 charges:
  - 6 non-physical $w_{\mu\nu}$,
  - 4 "magnetic" b_{μ},
  - 10 true: momentum p_{μ} and angular momentum $j_{\mu\nu}$.
- In Schwarzschild spacetime we don't have well-defined global charges (because we don't have adequate CYK tensors), we have only instantaneous charges,
- Charges can be used to glue strong field of a BH with a weak field outside,
- or to replace a strong field with a small particle that has adequate charges,

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Thank you for your attention