

Limit cycles of effective theories

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A simple example is used to show that renormalization group limit cycles of effective quantum theories can be studied in a new way. The method is based on the similarity renormalization group procedure for Hamiltonians. The example contains a logarithmic ultraviolet divergence that is generated by both real and imaginary parts of the Hamiltonian matrix elements. Discussion of the example includes a connection between asymptotic freedom with one scale of bound states and the limit cycle with an entire hierarchy of bound states.

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I. INTRODUCTION

A model [1] invented in the front form of Hamiltonian dynamics [2, 3] has been used in its generic matrix version to argue [4] that cyclic dependence of coupling constants on cutoffs in renormalization group (RG) procedure [5] may actually be commonplace for quantum mechanical systems whose Hamiltonians require renormalization. In independent studies, many examples of physical few-body systems with short range interactions and large scattering length have been considered in which a cycle structure may occur with universal features [6]. In the theory of three-body systems in nuclear [7] and atomic [8] physics, the limit cycle is associated with the existence of a sequence of bound states with binding energies forming a geometric series. The cycle has also been discussed in the context of the Bose-Einstein condensation [9] and superconductivity [10]. Even the elementary non-relativistic Hamiltonian for one particle in the potential $\sim 1/r^2$, when properly regulated, exhibits the cyclic behavior [11]. Concerning particle theory, the cyclic behavior of effective nuclear interactions could occur if the masses of π -mesons, which influence the shape of nuclear potentials, were increased by about one third. Since the masses of π -mesons can be linearly related to the masses of up and down quarks using the Gell-Mann-Oakes-Renner relation, the nuclear theory suggests that the masses of light quarks in the Standard Model are near the range where QCD may develop an infrared (IR) limit cycle [12, 13]. In order to determine if the IR limit cycle does indeed occur in QCD, the binding mechanism for quarks and gluons will have to be understood much more precisely than it is so far. While the mechanism of binding in just three-body systems has fascinated researchers for many years [14, 15] and a large body of literature exists on the subject [16], studies of the RG limit cycle appear to be in their infancy [17]. Quantum three-body calculations that trace the dependence of coupling constants on cutoffs tend to be complex [18]. Thus, the simple matrix example [4] provides useful insights concerning universality in Hamiltonian quantum mechanics with RG limit cycle [19].

So far, all studies of limit cycles known to the author

have been carried out in one of two ways. The first way, reminiscent of the concept of charge renormalization in quantum electrodynamics [20], is to regulate a diverging model with a cutoff and solve for some observable, such as a scattering amplitude or a bound-state energy, which is postulated not to depend on the cutoff. When the cutoff parameter is varied, it is possible to keep the observable fixed if instead a coupling constant is allowed to vary (there may be more than one constant to vary). Limit cycle means that the coupling constant is a periodic function of the cutoff parameter when the latter increases to infinity. Most of the quoted literature discusses examples of a limit cycle in this way. But if it is known that a theory being considered is only applicable to phenomena of a limited range of scales, the limit of infinite cutoff becomes logically questionable. The limit may not exist if the required coupling constants diverge in or on the way to the limit. The question of how to guarantee that all observables of interest are independent of a changing cutoff when it is finite (instead of being infinite) is hard to answer proceeding along the first way (unless one deals with a model in which the introduction of a small number of finite cutoff-dependent coupling constants is sufficient to completely eliminate the cutoff from equations that describe the model).

The second way is to apply Wilson's renormalization group procedure to a diverging model with a cutoff and compute the number of required constants and their shapes as functions of the cutoff [21, 22]. The required number of constants may be very large, or infinite, which is a typical situation for finite cutoffs (see below). The calculation involves a reduction of the number of degrees of freedom that are explicitly kept in the dynamics, and an evaluation of the change of the dynamics due to the reduction. Precise calculations can be done using discrete degrees of freedom and the number of the relevant degrees of freedom can be kept finite. The boundary of the set of the retained degrees of freedom plays the role of a new finite cutoff that is much smaller than the cutoff of the initial regularization (in the case of fixed-point criticality, finite cutoffs only mean that the system can be described by the same model at all scales). The finite cutoff defines the domain of an effective theory. If the calculation is precise, then by construction the finite

boundary cannot have any effect on the solution and can be set arbitrarily. However, the finite boundary limits the effective theory to a small space where the only source of dependence on the initial regularization cutoff is the explicit, computable dependence of the matrix elements of the effective-theory Hamiltonian on the initial cutoff. Using this information, it is possible to identify the changes in the setup of the initial regulated model that guarantee that the Hamiltonian matrix elements in effective theories with finite cutoffs do not depend on the initial cutoff. All predictions of such effective theory are then independent of the initial cutoff. It may happen that the presence of new terms, called counterterms, amounts to a replacement of the initial coupling constants with the new ones that depend on the initial cutoff. In principle, the outcome of the procedure is a family of effective theories with finite cutoffs and predictions that are guaranteed to be independent of the finite cutoffs. If a complete theory must contain many constants depending on the finite cutoff, additional conditions, such as the requirement of symmetries [23], may lead to finite correlations among otherwise unconstrained finite constants in the effective theories. One can consider the limit of an infinite number of degrees of freedom by simultaneously increasing the initial cutoff and using the RG transformation over an increasingly large range of scales to obtain effective theories with the same size of finite cutoffs. The limit cycle is then the ultimate RG orbit on which the effective coupling constants move indefinitely in the case of the infinite number of initial degrees of freedom. To approximate the limit cycle with increasing precision when the number of full cycles in the RG procedure increases, the initial regulated model must be set up in such a way that all unwanted terms that cause departures from or impede approach to the limit cycle in the RG procedure are eliminated [19].

This article concerns a new way to describe the cyclic behavior of effective theories, closely related to but different from the two ways mentioned above. Namely, the Hamiltonian matrices of effective theories are calculated using a renormalization group procedure called similarity [24, 25]. In the new procedure, no states are eliminated from the effective dynamics. Instead, the basis states are changed in such a way that the effective interaction Hamiltonian can dynamically couple only those basis states whose kinetic energies differ by less than an arbitrarily prescribed scale λ . This scale plays the role of a new renormalization group parameter. λ varies from infinity in the initial regulated model down to a finite value in an effective theory. When the initial regularization is being relaxed, the Hamiltonian matrix evaluated in the effective basis corresponding to a finite λ can lead to a dependence of perturbatively calculated observables on the initial regularization only through the explicit dependence of the Hamiltonian matrix elements on the initial cutoff. This explicit dependence is used to find the counterterms in the Hamiltonian of the initial model. Once the counterterms are found, the effective

theories are parametrized by finite values of λ and all have the same predictions that are independent of the initial regularization. The dependence of the effective Hamiltonian matrices obtained using the similarity RG procedure on λ is described in the following sections in the case of the same generic example that was previously studied in Refs. [4] and [19] using other methods.

The similarity procedure requires a generator (see next section) and details of the cyclic behavior of effective theories depend on how the generator is chosen. Since the main features of the cycle that are related to the spectrum of a renormalized model are not sensitive to the choice of the generator, variations in the cyclic behavior due to variations in the similarity generator will not be discussed in any detail. The explicit discussion in this article is focused on the case of a very elegant generator taken from Wegner's flow equation for (partial, in the case of a degenerate spectrum) diagonalization of Hamiltonian matrices [26, 27]. Similarity RG procedures using other generators may have better convergence properties in perturbative calculations [28], but this is of no significance to a qualitative understanding of the limit cycle whose description here is based on a non-perturbative numerical solution. The model considered here does not have degenerate eigenvalues and Wegner's generator leads to complete diagonalization when the RG procedure is carried out down to energy scales below the smallest eigenvalue.

Wegner [29] has recently pointed out that differential equations similar to his had been considered earlier for various purposes (other than a RG procedure) by mathematicians [30, 31, 32]. The study described here is limited to the context of RG procedure [24, 25]. The study shows that the limit cycle behavior of effective theories resembles in part the behavior of asymptotically free ones. Since the similarity approach can be applied to gauge theories (e.g., see [33] and later literature on the light-front formulation of QCD), the resemblance observed here in the simple model is also of interest in particle physics.

The model studied here is so simple that the cyclic dependence on λ of a large number of matrix elements of effective Hamiltonians can be approximately described in terms of just one function of λ . This function is called the renormalized coupling constant, g_λ , because it corresponds to and is closely related to renormalized coupling constants in effective Hamiltonians of quantum field theories. The main result is that each and every RG cycle consists of a sequence of gradual changes of g_λ , like in asymptotically free theories, followed by a sudden large change that brings the coupling constant back to its initial value in the cycle. In principle, this pattern repeats itself indefinitely when λ changes indefinitely. Each new cycle corresponds to a new bound state whose binding energy is on the order of λ at which the sudden change occurs.

The paper is organized as follows. Section II describes the model and produces the cyclic behavior of g_λ . Section III explains some elements of the mechanism of change

that g_λ exhibits in every cycle. Section IV concludes the paper with comments concerning implications of the model study.

II. MODEL

The model Hamiltonian is constructed for the purpose of qualitative understanding of the logarithmic similarity RG effects in the presence of bound states, which is a typical situation in the case of RG limit cycles. In order to construct a model that can be solved precisely, each and every energy scale is represented by just one state. The energy scale is defined by the spectrum of a Hamiltonian H_0 whose eigenvalues are assumed to be given by the formula $E_n = b^n$, where $b > 1$ and n is an integer. This means that the spectrum of H_0 is equally spaced on the logarithmic scale with step $\ln b$ (the unit of energy is set to 1). The corresponding eigenstates, denoted by $|n\rangle$, form an orthonormal basis, $\langle m|n\rangle = \delta_{mn}$. The interaction Hamiltonian in the model, H_I , is defined by its matrix elements in this basis.

Building models with such discrete basis is not a new idea [21, 22]. It is also a starting point for consideration of the case of $b \gg 1$. The case of large b is not accurate physically but it can further simplify the mathematics. There may exist complex physical systems where one can gain new insights by considering the case of very large b . For example, such strategy was instructive in the case of the Kondo problem [34], although quantitative numerical results were obtained for b around 2. There is hope that some way can be found to make use of a large b in studies of light front theory [33]. There, one also needs to find ways to drastically reduce the number of discrete degrees of freedom in dimensions other than the energy dimension.

The initial model Hamiltonian, $H = H_0 + H_I$, is defined by its matrix elements [4]

$$H_{mn}(g, h) = (E_m E_n)^{1/2} [\delta_{mn} - g - i h s_{mn}]. \quad (1)$$

For $m = n$, $\delta_{mn} = 1$ and $s_{mn} = 0$. For $m \neq n$, $\delta_{mn} = 0$ and $s_{mn} = (m-n)/|m-n|$. The interaction Hamiltonian, $H_I = H - H_0$, vanishes when $g = h = 0$.

The Hamiltonian of Eq. (1) has a generic ultraviolet structure with a logarithmic divergence to which both real and imaginary parts of the matrix elements contribute. Initial regularization is imposed by the cutoff $E_n \leq \Delta$. (In [4], the same cutoff was denoted by Λ . The notation is changed to avoid confusion with λ to be introduced below.) If one sets $\Delta = b^N$, the cutoff means $n \leq N$ and a logarithmic divergence in Δ means a linear divergence in N . When $h = 0$, the perturbative algebraic similarity procedure [24] produces interaction Hamiltonian with matrix elements of the form $g_\Delta (E_m E_n)^{1/2}$. This means that for $h = 0$ the counterterm that removes the dependence on $\Delta \rightarrow \infty$ from the model eigenvalues is obtained by replacing the constant g with a cutoff-dependent coefficient g_Δ . In fact, g_Δ exhibits asymptotic

freedom as a function of Δ when $h = 0$. When $h \neq 0$, g_Δ exhibits a limit cycle behavior (or chaos) [4]. h does not depend on Δ .

The similarity procedure yields a family of effective theories with Hamiltonian matrices labeled by λ . The family is described by a solution to the equation of the form

$$\frac{d}{d\lambda} H_\lambda = [F_\lambda\{H_\lambda\}, H_\lambda], \quad (2)$$

where $F_\lambda\{H_\lambda\}$ denotes the generator of the similarity transformation. The initial condition is set at $\lambda = \infty$ (in a numerical study, one can work with inverse of λ and start from 0, or the infinity is replaced by any number much greater than Δ). In the model studied here, H_∞ is given by H in Eq. (1) with g replaced by g_Δ . The case with $h = 0$ has been originally studied using Wegner's form of the generator $F_\lambda\{H_\lambda\}$ in [35, 36]. Writing the generator $F_\lambda\{H_\lambda\}$ in terms of its matrix elements as

$$\langle m|F_\lambda\{H_\lambda\}|n\rangle = f_{mn}(D_m - D_n)H_{\lambda mn}, \quad (3)$$

where $D_m = H_{\lambda mm}$ is the diagonal matrix element number m of H_λ , one obtains the following set of equations for all matrix elements of H_λ ,

$$\begin{aligned} \frac{d}{d\lambda} H_{\lambda mn} = \\ \sum_{k=M}^N [f_{mk}(D_m - D_k) + f_{nk}(D_n - D_k)] H_{\lambda mk} H_{\lambda kn}. \end{aligned} \quad (4)$$

The lower bound on the summation range, M , is an additional parameter introduced here to simplify numerical analysis of the low energy region. It sets an infrared bound on the lowest energy included in the model, $E_M = b^M$. The lower bound is very small for large negative values of M . The actual size of M is irrelevant to almost all aspects of the limit cycle behavior discussed here. The only exception are practical numerical issues and the range of λ over which one can easily observe the cycle using personal computers. M was set to -25 in all numerical examples discussed below.

Wegner's flow equation [26] is obtained for $f_{mn} = dl/d\lambda = -2/\lambda^3$. This choice amounts to the change of the RG variable λ to Wegner's parameter $l = 1/\lambda^2$.

The behavior of $(N - M + 1)^2$ matrix elements of H_λ as functions of λ results from solving the coupled nonlinear differential Eqs. (4). Numerical experiments showed that in order to clearly discern logarithmic effects for small coupling constants g_Δ and h (on the order of 0.1 or smaller) using a contemporary personal computer for less than an hour, one needs about 30 basis states with b on the order of 2. This gives about 1000 functions to trace. This number is sufficient for finding dominant features in the similarity RG evolution of the coupling constant because the behavior of the entire solution can be qualitatively understood by writing (f_{mn} disappears from the formula in the case of Wegner's equation)

$$H_{\lambda mn} = \sqrt{E_m E_n} A_{mn} \exp[-(E_m - E_n)^2/\lambda^2], \quad (5)$$

with matrix A_{mn} of the form

$$A_{mn} = \delta_{mn} - g_\lambda - ihs_{mn}. \quad (6)$$

The only term depending on λ is the coupling constant g_λ ; h remains constant.

Equation (6) is not precise. For example, using perturbation theory in the coupling constant g_Δ , one can find other terms in A_{mn} , such as $-2g_\Delta^2(E_m - E_n)^2/\lambda^2$. In fact, one also finds other functions of the subscripts m and n and the width parameter λ , and none of these structures is present in Eq. (6). Numerical calculation shows (see below) that these structures combine to produce the behavior of matrix elements of H_λ that is qualitatively captured by Eqs. (5) and (6). All matrix elements can be described using variables $q_{mn} = (E_m - E_n)/\lambda$. When λ decreases, q_{mn} for fixed m and n increases and the corresponding matrix element falls off. The dominant fall-off pattern is approximately reproduced by the simple Gaussian function that multiplies A_{mn} in Eq. (5). Although for matrix elements that have q_{mn} much larger than 1 the details of A_{mn} are not right and the Gaussian fall-off factor in Eq. (5) is not exact, the off-diagonal matrix elements with large q_{mn} (large changes of energy in comparison to λ) become equivalent to zero anyway. The simple Gaussian factor provides the same result. The diagonal matrix elements D_m with m corresponding to $E_m \gg \lambda$ are equal to the eigenvalues $\omega_m \gg \lambda$ of the initial H . The RG evolution of the matrix elements with $q_{mn} \sim 1$ was calculated numerically. Eq. (6) is accurate for matrix elements with $q_{mn} \ll 1$. The smaller q_{mn} the better the accuracy of Eqs. (5) and (6). The key question here is how g_λ depends on λ .

The best way to explain the behavior of g_λ is to use a generic example. On the basis of numerical experiments with various choices for the parameter b , coupling constants g_Δ and h , ultraviolet cutoff Δ , and infrared cutoff E_M , one representative case was selected that clearly displays key features of the cycle of the coupling constant g_λ in effective theories. The set of parameters includes $b = 4$, $g_\Delta = 0$, $h = \tan \pi/5$, $N = 16$, and $M = -25$. The value of b was selected as a result of compromise between making it large enough for a qualitative analysis (see next section) to be sufficiently precise for numerical confirmation, and keeping it small enough for the entire range of λ to remain within bounds set by requiring at least four digits of numerical accuracy from 4th order Runge-Kutta integration procedure for relevant matrix elements of H_λ . $g_\Delta = 0$ as an initial condition was selected to keep the calculation as simple as possible. Small values of g_Δ will be discussed later in connection with asymptotic freedom. The selected parameters are similar to those used in Refs. [35, 36] that analyzed the case with $h = 0$ (asymptotic freedom), to facilitate comparison.

Once the initial Hamiltonian is chosen, the similarity RG transformation is fully determined by Eq. (2). The running coupling constant is defined from the full solution through

$$g_\lambda = 1 - H_{\lambda MM}/E_M, \quad (7)$$

which is analogous to the definition of the fine structure constant in the Thomson limit. The continuous line in Fig. 1 displays the resulting g_λ . The broken line results from an approximate formula to be discussed in the next section.

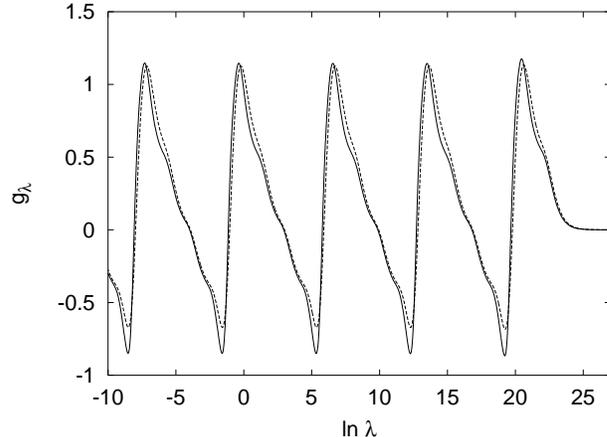


FIG. 1: The solid line displays the cycle. The broken line is an approximate result explained in Section III.

It is clear from Fig. 1 that the effective coupling constant oscillates with period $r = b^p$, where $p = \pi/\arctan(h) = 5$. For example, one can read the horizontal distances between successive points on Fig. 1 that correspond to $g_\lambda = 0.5$ and divide them by $5 \ln b$. From the numbers that were used to draw Fig. 1, one obtains (in the order from the largest to lowest value of $\ln \lambda$ in the figure): 1.0326, 1.0032, 0.9996, 0.9999, 0.9995, 0.9998, 0.9994, 0.9998.

It is also clear that the pattern of change of effective theories in a single cycle has two characteristic rates. Namely, the coupling constant changes gradually over almost the entire cycle and then suddenly jumps back to its initial value. This is a generic feature. It was observed numerically for all combinations of the parameters $1.1 < b < 100$, $2 < p < 42$, and coupling constants $g_\Delta < 1$, with no indication of disappearing outside this range. For example, if $p = 15$, i.e., for $h = \tan(\pi/15)$, the cycles are 3 times longer than in Fig. 1, but the jump takes about the same distance along the horizontal axis. For large p that considerably exceeds $N - M + 1$ and corresponds to an extremely small skew symmetric imaginary part in the interaction Hamiltonian, the cycle is so long that one can hardly distinguish the cyclic behavior of the coupling constant in the model with $N = 16$ from asymptotic freedom (see Fig. 3 in the next section). The next section discusses the mechanism of the cyclic change and shows how the cases of limit cycle and asymptotic freedom are connected.

III. THE CYCLIC CHANGE OF g_λ

The pattern of incremental changes followed by a sudden big jump that reverses the result of the gradual changes and starts a new period of incremental variation, is a characteristic feature of the model. The pattern is of interest by itself and may provide useful analogies in studies of other systems that display such behavior.

Matrix elements of the Hamiltonian H_λ can be written in the spectral form,

$$H_{\lambda mn} = \sum_{k=M}^N \omega_k \psi_{km}(\lambda) \psi_{kn}^*(\lambda), \quad (8)$$

where ω_k is the k -th eigenvalue of the model and ψ_{km} denotes m -th component of the corresponding effective wave function. The eigenstates are numbered from M to N in the order in which their eigenvalues appear on the diagonal when the similarity procedure is carried out all the way down to $\lambda = 0$. The wave functions are normalized by the condition $\sum_{m=M}^N |\psi_{km}(\lambda)|^2 = 1$. The eigenvalues are independent of λ .

The spectral decomposition implies through Eq. (7) that

$$g_\lambda = 1 - \frac{1}{E_M} \sum_{k=M}^N \omega_k \mathbf{a}_k(\lambda), \quad (9)$$

where $\mathbf{a}_k(\lambda) = |\psi_{kM}(\lambda)|^2$ is the probability of the lowest energy component in the eigenvector number k in the effective basis corresponding to λ . Taking into account that in the flow of H_λ towards small λ large eigenvalues appear one after another on the diagonal and off-diagonal matrix elements go to zero, one can conclude that the numbers $\mathbf{a}_k(\lambda)$ are reduced to zero one after another starting from large k . The eigenvector components are expected to vary with λ as the interaction does, and the approximate Eq. (5) suggests that $\mathbf{a}_k(\lambda) \sim \mathbf{a}_k(\infty) \exp[-2(E_k - E_M)^2/\lambda^2]$. A simple further consideration provides some insight concerning more precise description of the behavior of $\mathbf{a}_k(\lambda)$ as a function of λ .

Namely, in the case of $N = M + 1$, one can solve Eq. (2) analytically and the solution says that the small components of the eigenvectors indeed fall off as a Gaussian function, but the rate is given by $\exp[-(\omega_k - \omega_M)^2/\lambda^2]$, i.e. the eigenvalues of H_0 are replaced by the eigenvalues of H . Thus, knowing eigenvectors and eigenvalues of H one can write

$$\mathbf{a}_k(\lambda) = c_k(\lambda) \mathbf{a}_k(\infty) \exp[-2(\omega_k - \omega_M)^2/\lambda^2]. \quad (10)$$

$c_k(\lambda)$ is an associated normalization factor. It is computed using the condition that the similarity transformation does not change the norm and assuming that the RG evolution of the effective wave functions is given by the simple formula

$$\psi_{km}(\lambda) = c_k(\lambda) \psi_{km}(\infty) \exp[-(\omega_k - \omega_m)^2/\lambda^2]. \quad (11)$$

Phases of the complex wave-function components $\psi_{km}(\infty)$ increase in steps with m for $m > k$ with a period equal to the number of the steps needed to obtain 2π . A similar pattern occurs in the three-body dynamics [18].

Eq. (8) with formula (11) for $\psi_{km}(\lambda)$ is now used to approximate the entire RG evolution of the effective Hamiltonian matrices obtained from solutions of Eq. (2): the effective wave functions evolve with λ through the exponential factors $\exp[-(\omega_k - \omega_m)^2/\lambda^2]$ and the normalization coefficients $c_k(\lambda)$. The fall-off pattern of the approximate Eq. (10) is compared with exact numerically computed behavior of the wave functions in Fig. 2 using the example of 5 states forming a cycle.

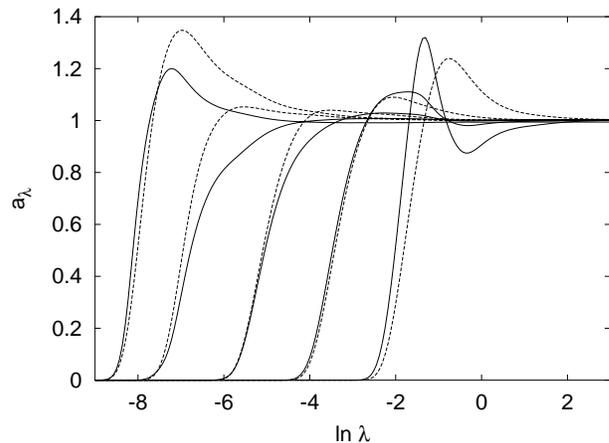


FIG. 2: Full lines show the λ -dependence of lowest energy components, i.e., $\mathbf{a}_\lambda = \mathbf{a}_k(\lambda)/\mathbf{a}_k(\infty)$, for 5 eigenvectors belonging to one cycle with $k = -6, -5, -4, -3$, and -2 , from left to right, respectively, and the broken lines display results of the approximate Eq. (10).

A comment is required concerning the order of numbering of eigenstates, which is related to why $b = 4$ was chosen in the example. The need for a comment stems from the intrinsic complexity of the eigenvalue problem and RG equations. A natural ordering of eigenvectors is found by tracing the sequence in which eigenvalues appear on the diagonal when λ is lowered. It turns out that the ordering depends on the size of b when the other parameters in the model are kept fixed. For example, there is a change in the ordering of eigenvalues between $b = 2.15$ and 2.20 in the case considered here. The entire Hamiltonian dictates where on the energy scale the negative bound-state eigenvalues are to appear. In other words, there is one negative eigenvalue per each and every quartet of successive positive eigenvalues but the modulus of the negative eigenvalue can wander across one of the positive eigenvalues on the diagonal when b is changed. $b = 4$ was chosen to stay clear of this feature near $b \sim 2.2$. But the interference among the evolving wave functions is not avoided entirely because the bound state wave functions spread coherently over the basis states and have more than one significant component. The interference is the origin of the difference be-

tween the qualitatively correct Eq. (10) and the actual dependence of wave functions on λ , visible in Fig. 2. In any case, all eigenvalues occur in sequences of powers of $r = b^p$. There are $p - 1$ such geometric sequences of positive eigenvalues and one such geometric sequence of negative eigenvalues. The sequence of negative eigenvalues is shifted on the logarithmic scale with respect to the sequences of the positive eigenvalues. Moduli of the negative eigenvalues are close to geometric averages of the neighboring positive eigenvalues. It is enough to find numerically where one of the moduli of negative eigenvalues fits in among the positive eigenvalues to locate moduli of all other negative eigenvalues on the energy scale (a numerical pattern loses accuracy near the cutoffs M and N).

Using Eq. (9) with input from Eq. (10), one obtains the broken line shown in Fig. 1. There is only a slight discrepancy between the broken line and the actual cycle, which follows from the complex interference pattern that is not fully captured. An additional benefit of the approximate formula is that it allows one to study behavior of the coupling constant for large N , $|M|$, p , and b approaching 1 with much less numerical effort than the Wegner equation itself. Although it is easy to underflow or overflow computer accuracy in studies of the logarithmic effects, some preliminary studies indicated that the effective coupling constant g_λ makes its rapid but smooth transition at the scale of binding for large p and b approaching 1 without any significant increase in size. The maximal value of the coupling constant may even decrease when b is lowered toward 1. The width of the transition region always appears to match the width of the bound-state wave function.

It is clear from Fig. 1 that the basic features of the cycle in behavior of g_λ as a function of λ can be described in the following way. The similarity RG flow eliminates interactions that change energy by more than λ and this forces eigenvectors to have their significant components squeezed on the H_0 energy scale to the region of size λ around the corresponding eigenvalues. The eigenvalues appear on the diagonal of H_λ one after another and the eigenvectors end up contributing only to the diagonal matrix elements, being eliminated from further evolution entirely. On the other hand, the elimination of p states of a full cycle brings the low-energy matrix elements, and therefore also the coupling constant, to the same values. A similar effect of elimination of states was shown already in Refs. [4, 19] using the Gauss elimination procedure as the RG transformation. In fact, elimination of p high-energy states produces a cyclic coupling constant behavior because

$$\sum_{k=n-p+1}^n \omega_k \mathbf{a}_k(\infty) = 0. \quad (12)$$

This identity is satisfied in the initial model Hamiltonian for $M \ll n \ll N$ with great accuracy independently of the RG procedure. The same pattern matters when

similarity changes $\mathbf{a}_k(\infty)$ to $\mathbf{a}_k(\lambda)$ and low energy components of high-energy states are eliminated one after another by the Gaussian weight that is characteristic of Wegner's equation. Since there is only one bound state per cycle, one term in the sum of Eq. (12) is a negative of all the others in a cycle together. This is why the incremental change in the coupling constant induced by RG evolution of $p - 1$ positive-eigenvalue eigenstates is undone by evolution of one negative-eigenvalue eigenstate.

The actual width of the rapid transition of g_λ as a function of λ must be related to the width of the bound-state wave function on the energy scale (in quantum field theory, the scale of eigenvalues of H_0 corresponds to the scale of kinematic momentum variables). For large b , only a few components of the bound-state wave function have significant size and the transition occurs over a few powers of b . When b is lowered, more components matter. The width of the bound-state wave function must determine the size of the rapid transition region because all significant components of the bound-state must be eliminated in order to produce the bound-state eigenvalue on the diagonal. Since a simultaneous rescaling of the parameter λ and all matrix elements of a cyclic Hamiltonian by r recovers the same array of matrix elements, the widths of successive bound states in the cycle also change like successive powers of r when λ is lowered. Thus, the width of the bound states that are eliminated when λ passes the scale of binding is constant on the logarithmic scale. The cyclic variation of g_λ must therefore occur with a constant width of the rapid-fall-off region on the logarithmic scale.

When p becomes very large, the incremental steps become very small in size in comparison to the whole cycle (for fixed b). But after the coupling g_λ grows to values much larger than h , its further change is governed by the positive spectrum practically independently of h . The same behavior of g_λ for λ near the scale of binding is obtained as in the case of asymptotic freedom in Refs. [35, 36], where $h = 0$. Fig. 3 illustrates how g_λ behaves in the two cases, both specified by $b = 4$, $N = 16$, $M = -25$.

The first case corresponds to a cycle with $h = \tan(\pi/50)$ and $g_\Delta = 0$. In this case, there happens to be just one bound state with energy $E = -7.644479 \cdot 10^{-6}$ (if N is increased to 66 and all other parameters are the same, another bound state with eigenvalue $E' = -9.690529 \cdot 10^{24}$ is created in the first case and $E' = 4^{50}E$). The second case corresponds to asymptotic freedom. The second case also has just one bound state with the same binding energy but $h = 0$ and $g_\Delta = 0.04000228$. In the second case, the bound-state is present due to a non-zero g_Δ . It is visible in Fig. 3 that as soon as g_λ evolves toward decreasing values of λ in the first case (limit cycle, solid curve) to values much larger than h , its further RG evolution toward still smaller values of λ is governed by the size of g_λ as in the second case (asymptotic freedom, dashed curve).

The rapid transition in the dependence of g_λ on λ when

λ passes through the scale of binding corresponds to the well-known infrared singularity in QCD (infrared slavery) that occurs when the momentum scale approaches Λ_{QCD} in perturbation theory. In the non-perturbative calculation in the model, the singularity appears smoothed because it is spread over the width of a bound-state wave function.

The asymptotically free model is analogous to a cycle of a very large range (very large p for fixed b , which means very small h) when g_Δ is set to a finite value in the initial Hamiltonian so that the RG evolution brackets the sudden-jump area on the energy scale due to a bound state. If $g_\Delta \neq 0$ in the cyclic model, the spectrum is slightly changed. It is shifted on the logarithmic scale and the range of the cyclic variation of the coupling constant is changed. The qualitative features for scales near the range of binding remain the same.

The key difference between the cycle and asymptotic freedom shows up when the coupling constant g_λ approaches 0. In the case of asymptotic freedom, the coupling tends to 0 as a fixed point when λ increases toward infinity. In the case of a cycle, when g_λ drops in size below h , the latter takes over in the dynamics and the evolution toward larger λ is altered.

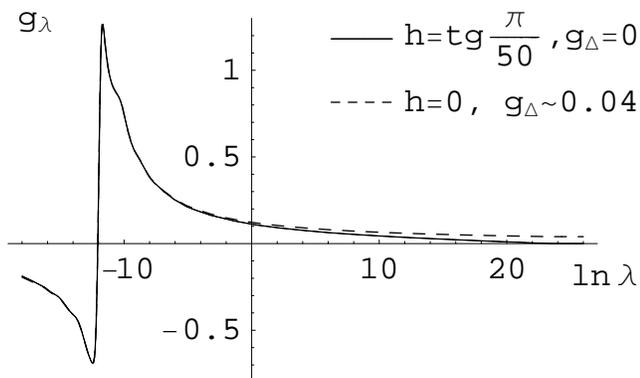


FIG. 3: Behavior of g_λ near the scale of binding in two cases: limit cycle and asymptotic freedom. The solid curve is obtained for the cycle with $p = 50$ and the dashed curve for asymptotic freedom with $g_\Delta \sim 0.04$ (see text for details).

IV. DISCUSSION

Limit cycles of realistic effective quantum theories that originate from models with divergences are difficult to study because of their complexity. The complexity can also obscure an underlying RG structure that may be close to a limit cycle, as occurs in a number of cases mentioned in the Introduction. The model discussed here is so simple that the cycle of effective theories in it can be exposed and analyzed without great effort. The study shows in a few steps how the similarity RG procedure may be set up for investigating limit cycles of effective theories. The cyclic behavior of the coupling con-

stant g_λ is found by solving Eq. (2) without calculating eigenvalues of the model with large cutoffs. The structure of counterterms is found using perturbative similarity RG transformation order-by-order. The transformation guarantees that eigenvalues of effective Hamiltonians with finite width λ are independent of λ .

The same similarity RG procedure that works in the case of a limit cycle also works in the case of asymptotic freedom. The procedure shows how the two cases are related. It also produces an approximate structure of solutions of effective theories in the form of Eq. (11). This approximate structure works in both cases and thus appears to be more generally valid than the models used here to identify it. The meaning of this result can be described in the following way.

Suppose a Hamiltonian for some physical system is proposed. Suppose that the proposed Hamiltonian renders results that are sensitive to arbitrary cutoffs that are not an intrinsic part of the proposal but a technical element of necessity: the proposed interactions spread states all over the space considered in the theory up to the cutoffs and the model without cutoffs, strictly speaking, does not exist. Matrix elements of the model Hamiltonian can be re-interpreted according to Eqs. (5) and (6) as corresponding to an effective theory. The re-interpretation consists of associating the basis states with energies known from experiments and multiplying the model wave functions for any given cutoff with a form factor analogous to the Gaussian factor present in Eq. (11). The form factor contains λ . For some finite value of λ on the order of observables of interest, the form factors in the new (effective) Hamiltonian and in the wave functions eliminate sensitivity to the arbitrary technical cutoffs. The effective theory needs adjustment of constants at some suitable value of λ .

In fact, a similar sequence of steps characterizes construction of effective theories in atomic, nuclear, or particle physics in which one introduces form factors to smear singular interactions. (The similarity RG evolution of g_λ in Hamiltonians of this article can be precisely related to the evolution of coupling constants in Hamiltonians of quantum field theories with form factors, e.g., see Ref. [37].) All that the similarity RG procedure provides here is a new context of looking at the construction of effective theories, especially in the case of limit cycles. In this context, when the energy *changes* described by the effective Hamiltonians increase, new bound states of smaller sizes (the size corresponds to $1/\lambda$) become a part of the effective dynamics. Of course, a bound state of a very small size cannot be resolved in processes limited to small changes of energy. There, it can only play the role of a new particle.

The close connection between two models: one with a limit cycle of a large scale period $r = \exp(\pi/h)$ and another one with asymptotic freedom, exposed here by the similarity RG procedure, provides further insights concerning a very large range of scales. It suggests that Hamiltonians with extremely small imaginary matrix el-

ements (extremely small coupling constants analogous to h after $\ln b$ is factored out) can produce a new generation of binding effects where asymptotic freedom alone predicts no new physics. The limit cycle structure thus appears to suggest a scenario in which a dynamical correlation can emerge across a very large range of scales due to an extremely small coupling constant in a well-defined theory; the smaller the coupling constant h , the larger

the scale period r . This is also a good reason for undertaking careful studies of limit cycles of effective theories using the similarity RG procedure and other means.

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