Articles

A Paradox of Two Space Ships in Special Relativity

Takuya Matsuda and Atsuya Kinoshita

A paradox in special relativity is proposed and discussed. Two spaceships start accelerating at the same time in an inertial frame S, in the same direction along the line joining them, experience the same acceleration for the same time duration, stop acceleration at the same time and reach the same velocity u, which is not too small compared with the light speed. Then, what is the distance between the two spaceships, after the steady motion is reached, as observed from S? Does it contract as is suggested by Lorentz transformation, or does it stay constant?

The answer is obvious: it stays constant. Interestingly, the distance, observed from a frame of the spaceships S′, does not stay constant but increases. The distance between two spaceships is not the same as the length of one spaceship.

The present authors discovered the interesting phenomenon that many physicists, including university professors who teach relativity, fail to understand the problem and insist that the distance between two spaceships should undergo Lorentz contraction. We solve this apparent paradox using Minkowski's space-time diagram, and explain possible pitfalls.

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1. INTRODUCTION

Imagine that a spaceship with a proper length L stays still at first in the inertial frame S, and then starts accelerating to reach a steady speed u. After the steady state is reached, the length of the spaceship observed from S contracts from L to L', where

$$L' = L\sqrt{1-(u/c)^2} \qquad (1)$$

This is the Lorentz contraction.

Now, let us imagine two spaceships of the same type, A and B, which stay still at first in the inertial frame S, the distance between the two spaceships being L. At t=0 these spaceships start accelerating in the same direction along the line joining A and B, undergo the same acceleration for the same duration, stop accelerating at the same time and reach a steady speed u, all viewed from S.

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Now we ask the question: "What is the distance between these two spaceships after they reach the steady speed *u*, observed from *S*? Is it *L* or *L′*? " The answer is obvious: it is *L*. The distance between two spaceships does not undergo Lorentz contraction contrarily to the length of one spaceship.

The present authors published papers presenting the above two spaceships paradox in a Japanese physics journal, although we do not refer the papers since they are written in Japanese. We discovered a very interesting phenomenon: many physicists, including university professors who appear to teach relativity, fail to understand the problem but instead claim that the distance should be *L′*. Some of them stick to the wrong answer and published papers criticizing us (in Japanese, therefore we do not refer them neither). Moreover, in order to give their wrong comprehension something to stand on, they presented lots of nonsensical arguments.

The aim of the present paper is to resolve this apparent paradox using a space-time diagram.

2. OBSERVATION FROM AN INERTIAL FRAME *S*

Figure 1 shows a schematic picture showing two spaceships, *A* and *B*, separated by *L* and flying in the same direction.

Fig.1: Two spaceships initially separated by a distance L in the inertial frame S start accelerating at the same time, undergo the same motion except for the starting points and reach the same final speed u. What is the distance between A and B as seen from S? Does it stay constant or does it Lorentz contract?

Figure 2 shows the space-time diagram of the world lines of the two spaceships observed from the inertial frame *S*(*x*, *ct*). The horizontal axis is the space coordinate *x* and the vertical axis is *ct*.

Before the acceleration, the two spaceships stay still on *S*, so the world lines of the two spaceships are $A \rightarrow A'$ and $B \rightarrow B'$. The distance between the two spaceships stays *L* before the acceleration.

In order to make the problem simple, let us assume that the time duration of the acceleration is infinitesimally small (but not zero), so that the world lines of the two spaceships are combinations of two straight lines. In Fig. 2 the event *B′* is just before the acceleration and *B″* is just after it. The world lines of the two spaceships after the acceleration are $A' \rightarrow A'' \rightarrow A'''$ and $B'' \rightarrow B'''$, where the line *A″B‴* is set to be parallel to *A′B″*. It should be noted that a distance on *S* should be measured at the same time on *S*. Therefore, the distance between two spaceships after the acceleration measured on *S* is *A′B″=A″B‴* (*=AB=L*). This is a conclusion drawn from a simple geometrical consideration only and has nothing to do with relativity.

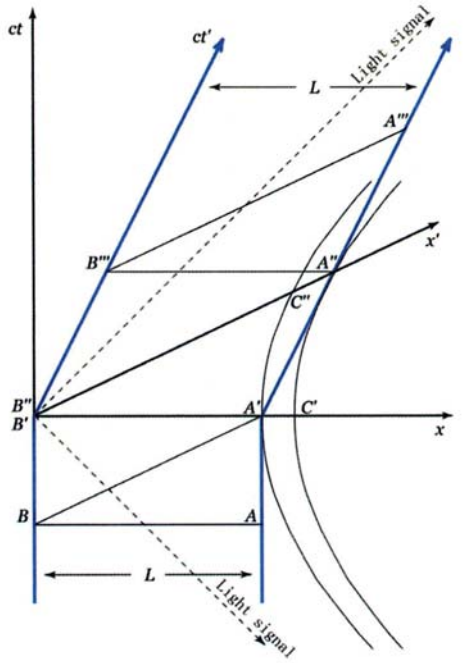

Fig. 2: World lines of two spaceships. Two spaceships stay still in the inertial frame S(x, ct) before accelerating. They accelerate instantly at t=0 and reach a steady speed u after the acceleration. The two spaceships stay still as observed from the inertial frame S′(x′, ct′), which is represented by the x′-ct′ coordinate. The dotted lines show the light cone.

One can conclude that the length of each spaceship may contract according to Eq. (1) but that the distance between the two spaceships should stay constant, as viewed from *S*.

3. EXPANSION OF THE DISTANCE BETWEEN TWO SPACESHIPS: OBSERVATION FROM *S′*

Now let us observe the situation from an inertial frame, *S′*, which is moving with speed *u* to the right. On this frame, *S′*, the two spaceships stay still after the acceleration. The frame *S′* is represented by *(x′, ct′)* in Fig. 2.

For an observer on *S′*, the events *A′* and *B′* do not occur at the same time, but *A″* and *B′* do. Two events are seen to occur at the same time on *S′*, if the line joining these two events is parallel to the *x′* axis. Therefore, *A′* occurs simultaneously with *B* rather than *B′(=B″)*, and *A‴* does with *B‴* , both to an observer on *S′*.

To the observer on S', the distance between two spaceships is $A''B''(=A'''B''')$, which is defined as L''. This length seems to be longer than $A'B''(=L)$, but we must be careful when making such a comparison. Indeed, the unit length on S and that on S' are not equal in this diagram.

In order to transform the length in S and S', let us use the following identity:

$$x^2 - (ct)^2 = (x')^2 - (ct')^2 = (L'')^2 \qquad (2)$$

This equation is visualized as the hyperbola, which is tangent to $A'A''A'''$ at A'' and tends to the light cone. The point, at which this hyperbola crosses with the x-axis, is denoted by C'. One may easily get $x'=L''$ by putting $t'=0$, and also $x=L''$ for $t=0$. Therefore, $B''C'=L''$ in this diagram. We may easily see that $B''C'>A'B''$: $L''>L$.

Quantitatively we can prove the following relation:

$$L'' = L \,/\, \sqrt{1-(u/c)^2} \qquad (3)$$

Therefore, we may conclude that the distance between two

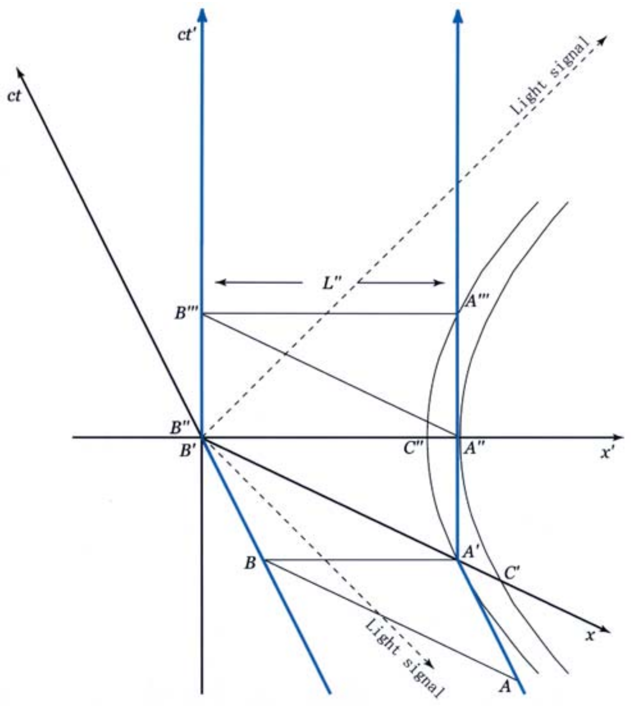

Fig. 3: The frame S′ is shown as a Cartesian coordinate system. The two hyperbolas and the light cone are unchanged compared with Fig. 2.

spaceships expands, when viewed from the frame S'.

4. LORENTZ CONTRACTION OF THE DISTACE BETWEEN TWO SPACE STATIONS

So far we showed that the distance between two spaceships should stay constant before and after the acceleration if viewed from S. We also found that the distance expands if viewed from S'. So, where is the Lorentz contraction?

Let us consider that the two spaceships are located at two space stations before accelerating. The distance between the two space stations is always L, of course, if viewed from S. The distance is seen to be L' in Eq. (1), if viewed from S'.

In order to see this more clearly, let us introduce a coordinate system in which x' and ct' are shown as Cartesian coordinates. Fig. 3 shows this. Note that all the notations in Fig. 3 are the same as those in Fig. 2.

From an observer on S', two spaceships as well as two space stations are moving to the left with speed u: A→A′ and B→B′. At the event A' the spaceship A stops by igniting a rocket motor, and later on at B' the spaceship B stops. Because of this time delay, the distance between the two spaceships increases for the observer on S'.

The distance between the two spaceships before the stop, which is the same as the distance between the two space stations, is viewed as $A'B$ on S'. This length $A'B$ is L'. In order to see this graphically, let us use the identity:

$$x^2 - (ct)^2 = (x')^2 - (ct')^2 = L^2 \qquad (4)$$

This equation can be represented as the hyperbola, which is tangent to AA' at A' and tends to the light cone. This hyperbola intersects with the x' coordinate at C''. We have $A'B'=C''B'$ $(=L)$, which is certainly longer than $A'B(=L')$. We may also prove that

$$C''B'(=L) < A''B'(=L'') \qquad (5)$$

Summarizing the above argument, the distance between two space stations contracts according to Eq. (1), if viewed from S'.

5. VIEW POINTS OF PILOTS OF THE SPACESHIPS

So far we discussed viewpoints of an observer either on S or S', both of which are inertial frames. Now let us consider the viewpoint of the pilots of the space ships, particularly that of the pilot of B.

For him the distance between the two spaceships is observed as $A'B'(=L)$ before the acceleration, but it increases suddenly to $A''B''(=L'')$ after the acceleration.

In other words, the spaceship A seems to warp from A' to A''. This is because an iso-time line jumps suddenly. For the pilot of B, A' is identical to A''.

This is an artifact due to our assumption that the acceleration occurs instantly. In reality the duration of the acceleration of B is not zero, although it can be infinitesimally short. Therefore, sudden jumps do not really occur.

6. WORLD LINE OF A SPACESHIP THAT PRESERVES THE PROPER DISTANCE

As was discussed above, the distance between two spaceships observed from a pilot point of view, which is called the proper distance, increases suddenly. Is there any world line that preserves the proper distance? There are many such world lines.

In order to present such an example, let us assume that the acceleration of spaceship B occurs instantly, but that of A occurs gradually. Fig. 4 shows such a world line: $A' \rightarrow C'' \rightarrow C'''$. Here the trajectory $A' \rightarrow C''$ is on the hyperbola given by Eq. (4) and $C'' \rightarrow C'''$ is a straight line. One can easily see that the proper length of the distance between two spaceships is preserved.

In Fig. 4, the original world line of $A' \rightarrow A'' \rightarrow A'''$ is also shown. The world line $A \rightarrow A' \rightarrow D'' \rightarrow D'''$ is nothing but that of the space station discussed in Section 4. We can see that

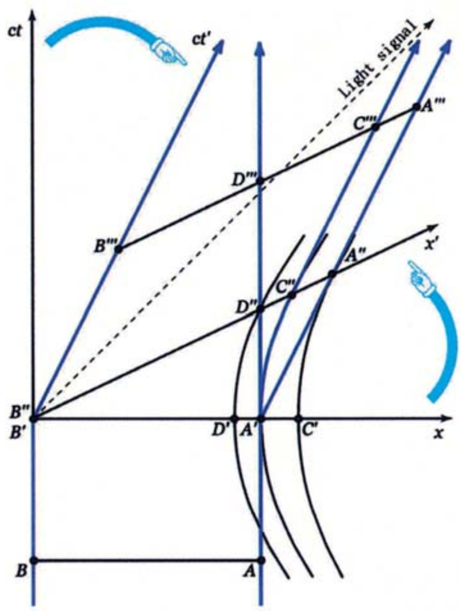

Fig. 4: World lines of a spaceship that preserves the proper distance. The world line of the head of the spaceship is $A \rightarrow A' \rightarrow C'' \rightarrow C'''$, while that of the tail is $B \rightarrow B' \rightarrow B'' \rightarrow B'''$.

$B''D''=B''D'(=L')<B''A'(=L)$.

7. GENERAL RELATIVISTIC VIEW POINT

The strange experience of the pilot on the spaceship B can be understood easily if a general relativistic viewpoint is introduced. For him gravity emerges suddenly at the time of the acceleration. The line element for an accelerating observer is

$$ds^2 = \left(1 + \frac{a}{c^2}x\right)^2 c^2 dt^2 - dx^2 - dy^2 - dz^2 \quad (6)$$

where a is the magnitude of acceleration/gravity. If a=0, we restore a Minkowski space-time.

For an observer on B, a distant clock seems to tick faster in proportion to the distance to the clock. This is due to the difference in gravitational potential. If we denote the proper time of the two spaceships as τ_A and τ_B, we have

$$\frac{d\tau_A}{d\tau_B} = 1 + \frac{a}{c^2}x \quad (7)$$

Based on this equation we may say that the clock on spaceship A seems to tick faster than that on B.

In the original case of the very short period of acceleration of spaceship A, the duration of acceleration is still shorter in A than B. Therefore, the spaceship A stops acceleration earlier than B, and the distance between them increases.

In the case of section 6, in which the proper distance between two spaceships is preserved, the acceleration of the spaceship A is adjusted to be smaller than that of B to compensate the faster tick of the clock on A. Denoting the acceleration of each spaceship as a_A and a_B, and if the following relation

$$a_A = \frac{a_B}{1+\frac{a_B}{c^2}L} \quad (8)$$

holds, then the proper distance between the two spaceships is kept constant as L.

8. MECHANISM OF LORENTZ CONTRACTTION: TWO SPACESHIPS JOINED BY A SPRING

World lines of A and B in the previous figures are those of two independent spaceships, but not those of the head and the tail of one spaceship. Why? The reason is that there is a body between the head and the tail of a spaceship, and this body exerts a stress.

In order to understand this, let us imagine two spaceships separated by L and joined by a spring, which is not very strong. If two spaceships accelerate in the same way, the distance between them increases at first viewed from the spaceships, as was discussed before. However, the distance begins to shrink because of the spring and settles to the original distance L viewed from the spaceships. In this way, the distance between two spaceships contracts following Eq. (1) viewed on S.

9. LIMIT LENGTH OF LORENTZ CONTRACTION

However, it should be noted that there is a limit length for which the Lorentz contraction really occurs. Let us imagine a huge spaceship with a proper length of 10 light years. Suppose it can reach $0.8c$ within a year, where c is the speed of light. According to Eq. (1), the length of the spaceship should become as small as 6 light years. However, in order for this to occur, some part of the spaceship should travel faster than c, and this is not possible. Therefore, such a long spaceship cannot contract according to Eq. (1). There must be some limit length for the Lorentz contraction really to occur. The limit is c^2/a, which can be inferred from Eq. (6) and (7).

10. DISCUSSION

After our Japanese papers and a few papers criticizing our argument appeared in the Japanese physics journal "Parity," we discovered that a very similar argument was discussed by Bell, and that it was met with similar criticism that we are [1]. This means that, unfortunately, many physicists did not, have not and still do not understand the real meaning of the Lorentz contraction even after almost 100 years of the introduction of special relativity by Einstein.

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11. REFERENCES

[1] John S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1989).

Fig. 5: Two spaceships A and B joined by a spring. The distance between A and B, observed from the spaceships frame, increases during the acceleration, and then contracts to its original distance L due to the stress of the spring. An observer on the inertial frame S measures the distance as L′, which is Lorentz contracted.